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Quantum entanglement as a quantifiable resource

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Quantum mechanical objects can exhibit correlations with one another that are fundamentally at odds with the paradigm of classical physics; one says that the objects are 'entangled'. In the past few years, entanglement has come to be studied not only as a marvel of nature but as a potential resource, particularly as a resource for certain unusual kinds of communication. This paper reviews two such uses of entanglement, called 'teleportation' and 'dense coding'. Teleportation is the direct, though not instantaneous, transfer of a quantum state from one object to another over a distance. Dense coding is the effective doubling of the information-carrying capacity of a quantum particle through prior entanglement with a particle at the receiving end. The final section of the paper presents various quantitative measures of entanglement and considers novel features that arise when entanglement is shared among three objects.

Keywords: entanglement; teleportation; dense coding; quantum communication

1. Entanglement and correlation

Entanglement is a special relationship between quantum particles that has no analogue in classical physics. Schrödinger said of entanglement, 'I would not call that *one* but rather *the* characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought' (Schrödinger 1935). The peculiarities of entanglement have been studied for decades, largely in connection with quantum non-locality and Bell's inequality (Bell 1964; Ballentine 1986), but lately there has been a resurgence of interest in the subject, stimulated in part by the discovery that entanglement, in addition to being interesting as a marvel of nature, could actually be *useful* in some ways. For example, entanglement is of crucial importance in the predicted operation of a quantum computer (Jozsa 1997), as other papers from this meeting will show, and also figures prominently in quantum cryptography (Ekert 1991; Bennett *et al*. 1992). Recently, it has been discovered that shared entanglement among several separated participants could allow them to answer more efficiently a question that requires them to pool their information (Grover 1997; Buhrman *et al*. 1997; Cleve & Buhrman 1997). In this paper I present the basic facts of entanglement, showing in particular how it differs from ordinary correlation, and I review two potential applications of entanglement in the field of communication: dense coding and teleportation. Finally, I discuss recent progress towards formulating a quantitative theory of entanglement. Most of the paper is a review of earlier work, mostly by other authors, though some new work is presented in the final section.

Before defining entanglement *per se*, it is helpful to lay out the basic laws of quantum mechanics as they apply to one of the simplest quantum systems, namely,

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the spin of a spin-$\frac{1}{2}$ particle such as an electron, proton, or neutron; I will use this as my standard example throughout the paper. The laws specify the possible quantum states of the spin, the possible measurements one can perform on it, and the possible transformations one can effect on it.

(1) *States*. The pure (that is, maximally specified) states of spin of a spin-$\frac{1}{2}$ particle correspond to the directions in three-dimensional space. Thus one can speak of the spin 'pointing' in some direction.

(2) *Measurements*. Only diametrically opposed directions can be distinguished from each other perfectly. Thus one can perform the 'up versus down' measurement, which distinguishes between the two vertical directions, or the 'right versus left' measurement, or any other measurement defined by a pair of opposite directions†. The probability of a given outcome of such a measurement is equal to $\cos^2(\frac{1}{2}\theta)$, where θ is the angle between the actual direction of spin and the direction associated with the measurement outcome. After a measurement has been performed, the measured particle does not retain its original spin state but rather has a state associated with the outcome of the measurement. For this reason one *cannot* perform on a single particle a measurement or sequence of measurements that would allow the observer to ascertain the original direction of the spin.

(3) *Transformations*. The most basic kind of transformation one can effect on a single spin is simply to rotate it by any angle around any axis‡. One does not need to know the initial direction of the spin in order to carry out the transformation.

Although it helps the intuition to view the spin states as directions in space as we have just done, a more fundamental description of a spin state is as a two-dimensional complex vector. In terms of the basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$, any spin state can be expressed as $a|\uparrow\rangle + b|\downarrow\rangle$, where a and b are complex numbers such that $|a|^2 + |b|^2 = 1$; the probabilities of 'up' and 'down' are given by $|a|^2$ and $|b|^2$. For example, the state 'pointing to the right' can be expressed as the superposition $(1/\sqrt{2})(|\uparrow\rangle + |\downarrow\rangle)$, and 'pointing to the left' is $(1/\sqrt{2})(|\uparrow\rangle - |\downarrow\rangle)$. The description in terms of spatial directions is based on a convenient mathematical connection (not to be detailed here) between directions in a three-dimensional real space and directions in a two-dimensional complex space (Park 1992).

We now consider the case of *two* spin-$\frac{1}{2}$ particles. One might think that in order to specify a pure state of two particles, it should be sufficient to specify the state of each individual particle. This would always be the case in classical physics, but it is not so in quantum mechanics. A general pure state of two spins can be written as

$$a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle, \tag{1.1}$$

where a, b, c and d are complex numbers such that $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$, and $|\uparrow\uparrow\rangle$, for example, is the state in which both spins are pointing up. Some states of the form (1.1) can be factorized into separate states of the individual particles via the tensor product $\otimes$ (e.g. $|\uparrow\rangle \otimes |\downarrow\rangle = |\uparrow\downarrow\rangle$). It happens that such factorization can be done only if $ad - bc = 0$, so that most states of the form (1.1) do not factorize. The non-factorizable states are called entangled and are characterized by

† There do exist more general 'non-orthogonal' measurements (Helstrom 1976; Peres 1990), but they still do not allow a perfect discrimination between directions that are not diametrically opposed.

‡ As with measurements, quantum mechanics allows more general transformations, usually involving a quantum interaction with another object (Kraus 1983), but we will not need such generalized transformations in this paper.

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the fact that even though the pair of spins taken as a whole has a definite pure state, neither spin has a pure state of its own. Note that the real quantity $|ad - bc|$ can be taken as a measure of the degree of entanglement; indeed we will see later that the accepted information-theoretic measure of entanglement is a function of this quantity. The notion of entanglement applies also to mixed states, that is, states that do not provide as complete a description of a system as quantum mechanics allows. We consider the case of mixed states in the final section of the paper.

Entangled states are extremely common in real life. The two electrons of a helium atom in the ground state, for example, have their spins entangled, and indeed any two quantum particles that are interacting with each other will typically be entangled. Entanglement is more interesting, though, when the entangled particles are well separated from each other, and this condition is more difficult to achieve. Separated but entangled photons have been generated by the successive emission of two photons from a single atom (see, for example, Freedman & Clauser 1972)), and by the splitting of a single photon into two by down-conversion (see, for example, Kwiat *et al.* 1995) as described at this meeting by Professor Zeilinger. Entangled spin-$\frac{1}{2}$ particles have been produced by the scattering of one proton off another (Lamehi-Rachti & Mittig 1976), and very recently the successful production of a pair of separated but entangled *atoms* was achieved, using a microwave cavity as a catalyst (Hagley *et al.* 1997). The mathematics of entanglement is essentially the same for all these physical realizations. For example, to translate our spin-$\frac{1}{2}$ expressions into the language of photon polarization, one simply interprets '↑' as 'vertical polarization' and '↓' as 'horizontal polarization'. The mathematical similarity derives from the fact that all the objects mentioned are effectively *binary* quantum systems, or *qubits*; that is, each admits no more than two orthogonal (i.e. perfectly distinguishable) states†. Entanglement also applies to more complex systems, but in this paper I confine my attention to qubits.

To illustrate the properties of entangled states, let us consider the particular entangled state $(1/\sqrt{2})(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$. If one were to make the 'up versus down' measurement on each of the two particles, then, reasonably enough, one would find that the outcomes always agree with each other, with equal probabilities of 'up up' and 'down down'. It is less obvious, but true, that one gets the same sort of correlation upon performing the 'right versus left' measurement on each particle. To see why this is the case, note that the states $|\uparrow\rangle$ and $|\downarrow\rangle$ can be expressed as $(1/\sqrt{2})(|\rightarrow\rangle + |\leftarrow\rangle)$ and $(1/\sqrt{2})(|\rightarrow\rangle - |\leftarrow\rangle)$, respectively, so that our entangled state $(1/\sqrt{2})(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$ can just as well be expressed (after a little algebra) as $(1/\sqrt{2})(|\rightarrow\rightarrow\rangle + |\leftarrow\leftarrow\rangle)$, which makes the correlation manifest. By similar algebra one can show that the state $(1/\sqrt{2})(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$, which one might at first think is physically equivalent to $(1/\sqrt{2})(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, actually exhibits an *anticorrelation* when measured along the 'right versus left' axis. Thus these two similar-looking entangled states are quite distinguishable.

The preceding paragraph illustrates one of the primary distinctions between quantum entanglement and ordinary correlation: entangled particles exhibit a correlation not just for one measurement but for a whole class of mutually exclusive measurements. Two coins may be correlated in the sense that both are heads or both are tails, but with coins there is no complementary measurement analogous to 'right

† An atom, of course, has many states, but in the experiment cited, it is arranged that only two of them have any substantial probability of being occupied.

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versus left' that one could perform instead of the heads–tails measurement. Another crucial difference between entanglement and classical correlation is this: in classical physics if the state of a composite system is maximally specified, i.e. if it is a pure state, then the state of each part of the system *must* be maximally specified as well. However, for an entangled state, as we have already mentioned, the components cannot be assigned pure states of their own even if the system as a whole is in a pure state. This is a radical departure from the paradigm of classical physics.

Finally, with entanglement there is a specificity of connection that does not apply to ordinary correlation. Two particles cannot become entangled just by having similar properties. This is one of the lessons of Bell's work, which showed that mere classical correlation could never explain all the effects predicted by quantum mechanics (Bell 1964; Ballentine 1986). To become entangled, a pair of particles must interact with each other somehow, whether it be directly or indirectly via a third quantum particle. In human terms, ordinary correlation is like enjoying the same books—two people do not have to have met each other in order to enjoy the same books—whereas entanglement is more like being married. This connectedness between entangled particles stands out perhaps more clearly when one considers the two potential applications, dense coding and teleportation, described in the following sections.

2. Dense coding

It is a plausible but non-obvious fact that the spin of an unentangled spin-$\frac{1}{2}$ particle can carry no more than one bit of information. Conveying one bit in such a particle is easy: the sender simply encodes the digit 0 as the 'up' state and the digit 1 as the 'down' state, and the receiver distinguishes these states by an 'up versus down' measurement, thus recovering one bit. What is not so obvious is that one cannot do better. For example, one could imagine encoding 00 as $|\uparrow\rangle$, 01 as $|\rightarrow\rangle$, 10 as $|\leftarrow\rangle$ and 11 as $|\downarrow\rangle$, thereby expressing two bits in a single spin. Of course those four states cannot be distinguished perfectly from each other, so that one cannot hope to convey a full two bits in this way, but it is not *a priori* foolish to imagine some clever measurement that could distinguish the four states well enough to give the receiver, say, 1.1 bits of information. As it turns out, a theorem proved in 1973 by Kholevo guarantees that no coding scheme, however clever, can be used to transmit more than $\log_2 n$ bits in a quantum particle that has exactly n orthogonal states (Kholevo 1973; Levitin 1969). For our spin-$\frac{1}{2}$ particles, $n = 2$, so that the theorem prohibits sending more than a single bit per particle.

However, this consequence of Kholevo's theorem applies only to an *unentangled* particle. As we are about to see, if the particle being transmitted is already entangled with a particle at the receiver's end, then it can be used to send a full two bits of information; that is, it can be used to send one of four possible messages (Bennett & Wiesner 1992). This effect is known as dense coding.

To set the stage for dense coding, the sender and receiver, conventionally called Alice and Bob, have to arrange in advance the sharing of an entangled pair of particles. For definiteness let us suppose that their shared pair is in the state $(1/\sqrt{2})(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$. Here the first symbol in each $|\ldots\rangle$ refers to Alice's particle and the second to Bob's. This sharing may be done long before Alice knows what message she wants to send. Thus we imagine Alice holding on to her particle while Bob is waiting with his particle at a distant location.

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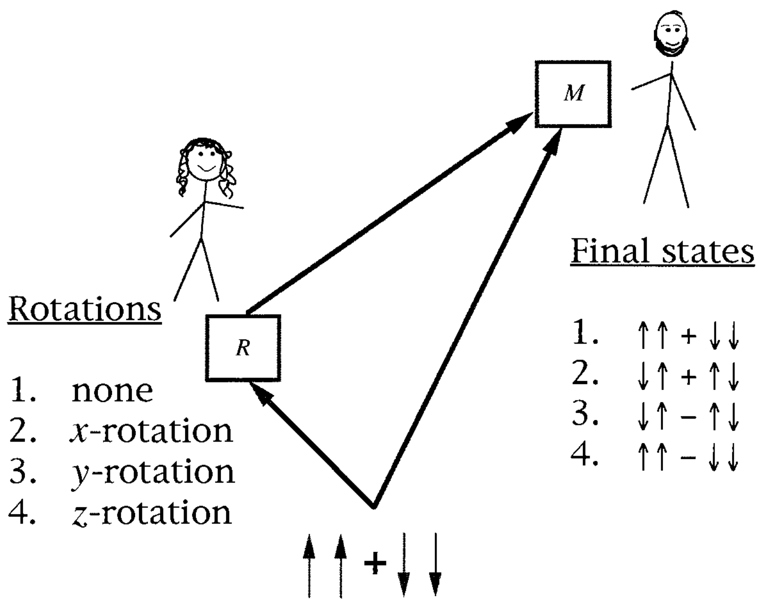

Figure 1. A spacetime picture of dense coding. Time goes upward in this picture: first an entangled pair is produced and shared; then Alice performs her rotation R, and finally Bob uses his measurement M to distinguish the four possible final states. (For simplicity the normalization factor $(1/\sqrt{2})$ has been omitted.)

At some moment Alice finds out which of four messages she wants to send. She has agreed in advance with Bob that each message will correspond to a particular operation that she will perform on her particle. After performing the operation, she will send her particle to Bob. Alice's four possible operations are the following: (i) do nothing; (ii) rotate the spin by 180° around the x-axis (i.e. the right–left axis); (iii) rotate by 180° around the y-axis (the other horizontal axis); and (iv) rotate by 180° around the z-axis. Although Alice acts only on her own particle, the effect of her action is to change the entangled state of the pair. The following table shows the final state associated with each of Alice's four possible actions.

	action	associated state
(i)	do nothing	$(1/\sqrt{2})(\vert\uparrow\uparrow\rangle + \vert\downarrow\downarrow\rangle)$
(ii)	x-rotation	$(1/\sqrt{2})(\vert\downarrow\uparrow\rangle + \vert\uparrow\downarrow\rangle)$
(iii)	y-rotation	$(1/\sqrt{2})(\vert\downarrow\uparrow\rangle - \vert\uparrow\downarrow\rangle)$
(iv)	z-rotation	$(1/\sqrt{2})(\vert\uparrow\uparrow\rangle - \vert\downarrow\downarrow\rangle)$.

(2.1)

The second of these, for example, follows from the fact that a 180° rotation around the x-axis changes $|\uparrow\rangle$ into $|\downarrow\rangle$ and vice versa. The last two are less obvious: the y-rotation changes $|\downarrow\rangle$ into $-|\uparrow\rangle$ and the z-rotation changes the sign of $|\downarrow\rangle$.

Once Bob possesses both particles, he now has the task of distinguishing among the above four states. In principle this distinction can be made, because the four states are mutually orthogonal. Thus Bob can determine which of the four possible messages Alice was trying to send, thereby gaining two bits of information as advertised. The whole procedure is summarized in figure 1.

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In practice, the measurement Bob is required to make, known as the Bell measurement, is very difficult and has in fact not yet been performed on any pair of particles. A somewhat less ambitious measurement has, however, been performed on pairs of photons, using ordinary optical components and careful detection of coincidences (Mattle *et al*. 1996). In this measurement, states (ii) and (iii) (or rather, the photonic counterparts of these spin states) are distinguished from each other and from the other two, but states (i) and (iv) are not distinguished from each other. To make a complete Bell measurement on two particles is by no means impossible and will probably be done in the next several years, but it will probably require an entirely different sort of apparatus (Cirac & Parkins 1994; Davidovich *et al*. 1994; Sleator & Weinfurter 1995).

Although with current technology it is difficult to imagine any situation in which it would be more economical to use dense coding rather than to use ordinary coding with twice as many particles, the fact that entanglement is capable of increasing the information-carrying capacity of a particle is of great theoretical significance. It means that local actions on a single quantum particle can faithfully express more information than could possibly be stored in the particle itself. The information is clearly being stored not in the particle but in the entanglement between two particles.

3. Teleportation

Whereas the aim of dense coding is to convey ordinary classical information, e.g. zeros and ones, through the medium of quantum particles, the aim of quantum teleportation is to convey quantum states themselves.

Consider the following problem. Alice has been given a single spin-$\frac{1}{2}$ particle whose spin state $|\psi\rangle$ she does not know, and she would like to convey this state to Bob. That is, at the end of the transmission, Bob should have a spin-$\frac{1}{2}$ particle whose state is $|\psi\rangle$. If there were no further restrictions, the simple solution would be for Alice merely to send her particle to Bob. However, we stipulate also that Alice does not know and cannot find out where Bob is, and this makes the problem much harder.

If Alice could determine her particle's state by measurement, then she could broadcast that information to all the places Bob might be and he could reconstruct a particle with the desired state. However, as we have said, it is impossible to determine an unknown spin state by any measurement; so this approach fails. Alternatively, if Alice could make many copies of the original particle, she could send these copies to all the places Bob might be. But quantum mechanics forbids the 'cloning' of an unknown quantum state (Wootters & Zurek 1982; Diekes 1982).

The solution is teleportation (Bennett *et al*. 1993), which, like dense coding, requires some prior preparation on the part of Alice and Bob. Knowing that this communication problem is going to arise, Alice and Bob, at some time when they are together, share an entangled pair. Let us suppose that they use the same entangled state as in the preceding section, namely, $(1/\sqrt{2})(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$. Each of them then carries his or her half of the entangled pair, in a suitcase as it were, as the two of them travel about. Later, when Alice finally receives the particle whose state she is to convey, she opens her suitcase and begins the teleportation procedure.

This procedure is in a sense the reverse of dense coding and is illustrated in figure 2. Alice starts with two particles: the one whose state $|\psi\rangle$ is to be teleported, and the one she has been carrying with her. On these two particles she performs the Bell

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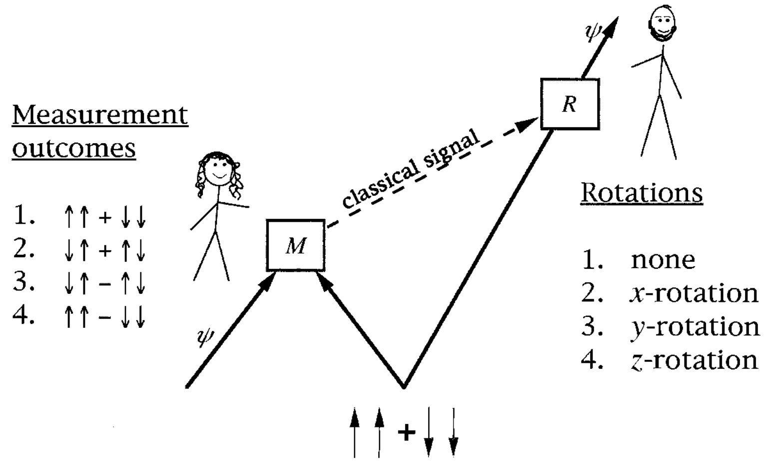

Figure 2. Teleportation. Alice performs a Bell measurement on her two particles and tells Bob the result. Then Bob performs the corresponding rotation on his particle.

measurement, which distinguishes among the four states listed in (2.1). She then broadcasts the outcome to Bob, who performs the corresponding rotation (according to the same table) on the particle he has been carrying with him. The laws of quantum mechanics then guarantee that the final state of Bob's particle is $|\psi\rangle$, so that the teleportation is complete.

Exactly how the laws of quantum mechanics predict this effect is shown explicitly in Bennett *et al.* (1993) and will not be repeated in full here. The essence of the calculation is as follows: the original state of the three particles can be expressed as a linear combination of four basic states. Each of these basic states describes Alice's two particles as being in one of the four states of (2.1) and Bob's particle as being in some corresponding spin state. When Alice's measurement is made, it picks out one of the four basic states of Alice's particles, and therefore also picks out the corresponding state of Bob's particle. As it happens, the state of Bob's particle is in every case related by a simple rotation to the original state $|\psi\rangle$ that was to be teleported. By performing that rotation, Bob brings his particle to the state $|\psi\rangle$. Note that the teleportation is not complete until Bob has performed this rotation, and he cannot know what rotation to perform until he learns the result of Alice's measurement. Since the signal from Alice is an ordinary classical signal, Alice and Bob cannot use teleportation to transmit a quantum state faster than light.

That quantum mechanics predicts teleportation in this sense is not controversial. Indeed the essential effect has now been observed experimentally (Bouwmeester *et al.* 1997; Boschi *et al.* 1998). Moreover, it is clear that no new mechanism needs to be invoked in order to explain the effect: the machinery of quantum mechanics provides all the mechanism one needs. Nevertheless, it is difficult to say in plain language precisely what is going on, and, at the risk of opening a notorious can of worms, I think it will be worth spending a few paragraphs struggling with this task.

First let us consider more closely the Bell measurement that Alice performs. In a sense she is asking her two particles which of the four entangled states of (2.1) expresses their condition. In actuality none of these states applies to those two particles: they have never interacted with each other in the past and are certainly not

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entangled with each other. Alice's measurement nevertheless effectively *forces* the two particles to choose one of these entangled states, just as an 'up versus down' measurement on a single particle forces one of the choices 'up' or 'down' even if the particle was originally pointing to the right. In the case of Alice's measurement, each of the four outcomes is equally likely, *regardless* of the spin state $|\psi\rangle$ to be teleported.

Let us suppose for definiteness that Alice's two particles happen to choose the last of the four possible outcomes, namely, the state $(1/\sqrt{2})(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$. Then quantum mechanics predicts that from the moment this choice is made, we may safely regard Bob's particle as being in the state $R_z|\psi\rangle$, where R_z is the 180°-rotation around the z-axis. By 'safely' I mean that in practice any testable predictions following from this assumption will be correct. This is a remarkable fact. For the whole time that Alice and Bob have been apart, Bob's particle has been sitting undisturbed in his suitcase. And yet once Alice's measurement has been performed and the outcome $(1/\sqrt{2})(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle)$ obtained, Bob's particle has apparently taken on the state $R_z|\psi\rangle$, even though $|\psi\rangle$ may not have been given to Alice until long after Bob's particle was sealed up. It is difficult to imagine how the state $R_z|\psi\rangle$ might have found its way to Bob's particle, and this is what makes the effect hard to talk about.

When speaking informally, we physicists often explain the effect by saying that when Alice's measurement is made, it collapses Bob's particle to a particular spin state at the same instant. But we know that this is not a correct way of speaking, if only because according to relativity theory, the notion of 'the same instant' is fundamentally ambiguous, depending as it does on the state of motion of the observer.

One relativistically acceptable account of the phenomenon begins by resisting the temptation to assign any definite spin state to Bob's particle until Bob has learned the outcome of Alice's measurement and has performed the appropriate rotation. On this view, the effect of Alice's measurement is not to collapse Bob's particle but rather to entangle it with everything associated with Alice's measurement, including the apparatus, the classical signal, and Alice herself. This entanglement persists until the signal interacts with Bob's particle (through Bob) and the latter finally achieves a pure spin state of its own. This way of speaking is in the spirit of the many-worlds view, in which macroscopic objects such as Alice and Bob are often entangled with the rest of the universe (DeWitt & Graham 1973).

An alternative model, which avoids such personal entanglements, is the following. The information in the state $|\psi\rangle$ comes to Bob by a kind of zigzag path in spacetime: it is reflected backward in time by Alice's measurement, and then is reflected forward again by the preparation of Alice's and Bob's entangled pair. However, the 'mirror' of Alice's measurement may be a distorting one, so that at the end Bob may have to correct for the distortion. He cannot know how to correct it until he learns what sort of 'mirror' was used, that is, until he learns the outcome of Alice's measurement (similar ideas have been expressed by Costa de Beauregard (1977), Cerf & Adami (1997), Bennett & Wiesner (1992), and Penrose (personal communication)). Though there is something very strange about a backward-in-time signal, at least we can say that this signal has a physical embodiment, namely, the particle in Alice's suitcase, whereas there is no candidate object that could embody an instantaneous signal transmitted at the moment of Alice's measurement.

Whatever one may think about causality or many-worlds, physicists agree on those aspects of teleportation that can be tested in practice. And in practice teleportation may turn out to be quite useful. One potential application is the transmission of

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quantum states through a noisy channel. Suppose Alice wishes to send the state of a spin-$\frac{1}{2}$ particle through a randomly fluctuating magnetic field, which would perturb the state in an unpredictable way. Then instead of sending the particle directly, she could do the following. First she creates a large number of perfectly entangled pairs and sends one member of each pair to Bob through the noisy channel. The noise will pollute the entanglement of each pair, but as long as the noise is not too great, there are methods by which Alice and Bob can distil, from the imperfectly entangled pairs, a smaller number of perfectly entangled pairs (Bennett *et al*. 1996*a*). Alice then uses one of these perfect pairs to teleport the state of her original particle.

One can also imagine other uses of teleportation (Bouwmeester *et al*. 1997; Boschi *et al*. 1998). Most likely, however, the effect will eventually turn out to be most useful in a context that we have not yet imagined.

4. Quantifying entanglement

If entanglement is a resource, one would like to know how much of that resource one possesses in any given situation. More generally, if entanglement is interesting enough to be studied seriously, we will need a quantitative theory of it. Much work has been done in the last few years on finding well-justified quantitative measures of entanglement. This section reviews this work briefly and then considers some of the new features that arise when the entanglement involves three particles rather than two.

The two-particle state

$$a|\uparrow\uparrow\rangle + b|\downarrow\downarrow\rangle, \tag{4.1}$$

with $|a| > |b|$, is not as entangled as it would be if a and b were equal; for example, the right–left correlation is not as strong. But to what *extent* is it less entangled? We would like to find a natural way of assigning a number to such a state indicating the degree of entanglement. For this purpose it is helpful to imagine the following scenario (Bennett *et al*. 1996*b*).

Alice and Bob are at fixed locations some distance apart and initially share no entanglement. Eventually they would like to share n pairs of particles, each pair being in the state $a|\uparrow\uparrow\rangle + b|\downarrow\downarrow\rangle$. They cannot accomplish this without the transmission of actual quantum objects—mere classical communication will not be enough—so let us suppose that Alice will send quantum particles to Bob. For example, Alice could prepare all n pairs in her own laboratory and send one member of each pair to Bob. This strategy would require the transmission of n qubits. But there is a more efficient strategy. Once Alice has prepared the n pairs, she can *compress* into a smaller number of qubits all the entanglement information embodied in the n particles she was going to send to Bob. If she does this properly and sends the smaller collection of qubits to Bob, Bob will be able to reconstruct a set of n particles that are entangled in the desired way with the n particles that Alice still holds. The maximum possible extent of this compression is known (Bennett *et al*. 1996*b*; Schumacher 1995; Jozsa & Schumacher 1994): asymptotically, as n becomes very large, the optimal compression factor is equal to the *entropy* of each member of a pair, which for the state $a|\uparrow\uparrow\rangle+b|\downarrow\downarrow\rangle$ is given by the formula

$$\text{entropy} = S = -(|a|^2 \log_2 |a|^2 + |b|^2 \log_2 |b|^2). \tag{4.2}$$

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That is, in order for Alice and Bob to share n pairs in the state (4.1), where n is large, it is sufficient that Alice send Sn qubits to Bob. (Note that $S \leqslant 1$.) We define the *entanglement* of any pure state of a bipartite system to be the minimum asymptotic number of qubits per pair that Alice needs to send to Bob in order to create many pairs in the desired state (Bennett *et al*. 1996*b*). Thus the entanglement of the state $a|\uparrow\uparrow\rangle + b|\downarrow\downarrow\rangle$ is equal to the entropy S.

By choosing the correct basis, one can write any pure state of two qubits in the form (4.1), so that equation (4.2) can be used to find the entanglement of any such state. The result can be expressed as follows (Hill & Wootters 1997; Wootters 1998). For the general state

$$|\phi\rangle = a|\uparrow\uparrow\rangle + b|\uparrow\downarrow\rangle + c|\downarrow\uparrow\rangle + d|\downarrow\downarrow\rangle, \tag{4.3}$$

the entanglement is

$$E = h[\tfrac{1}{2}(1 + \sqrt{1-\tau})], \tag{4.4}$$

where $h(x) = -(x \log_2 x + (1-x)\log_2(1-x))$ and the 'tangle' τ of the state $|\phi\rangle$ is given by

$$\tau = 4|ad - bc|^2. \tag{4.5}$$

The tangle of a state of two qubits is a number between zero and one, as is the entanglement E, and E is a monotonically increasing function of τ. Thus either E or τ can be taken as a measure of entanglement; τ is a simpler function of the vector components, but E has the immediate physical interpretation described above.

We now turn our attention to *mixed states* of two spin-$\frac{1}{2}$ particles. One describes a quantum system by a mixed state either because one does not know which pure state to use or because the system in question is entangled with something else, in which case it does not have a pure state of its own†. A mixed state is called entangled if the state cannot come to be shared by two spatially separated observers without the transmission of quantum objects (see, for example, Peres 1996; Horodecki *et al*. 1996). This criterion has been widely accepted by researchers in the field. However, there appears to be no unique measure of the *degree* of entanglement of an entangled mixed state. Three distinct information-theoretic measures of entanglement have been proposed for a general mixed state of a bipartite system.

(1) The *entanglement of formation* is the asymptotic number of qubits per pair that Alice needs to send to Bob in order to create a large number of pairs in the given state (Bennett *et al*. 1996*c*).

(2) The *relative entropy of entanglement* measures the distinguishability between the given state and the nearest unentangled state (Vedral *et al*. 1997*a*, *b*).

(3) The *distillable entanglement* measures the efficiency with which a large number of pairs in the given state can be converted into pure completely entangled qubit-pairs using only local operations and classical communication (Bennett *et al*. 1996*a*).

For pure states all these measures coincide. For mixed states the relationships among the three quantities have not yet been worked out. It is still conceivable that they are all the same, though the evidence suggests that they are different and are related by the following inequalities (Vedral & Plenio 1998):

$$\text{entanglement of formation} \geqslant \text{relative entropy} \geqslant \text{distillable entanglement}. \tag{4.6}$$

† A mixed state is not described by a state vector but rather by a density matrix (Peres 1995), which can be expressed as a probabilistic mixture of pure states.

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Clearly much work remains to be done towards understanding these relationships. It is also interesting to ask whether, given a particular mixed state, there is a simple formula for each of these measures. Some encouraging evidence along these lines has been found in the case of a pair of qubits (Hill & Wootters 1997; Wootters 1998), but at present our ignorance on this subject is much greater than our knowledge.

The three quantities just described are all information-theoretic measures of entanglement, in the sense that they express asymptotic efficiencies in the spirit of Shannon's classical coding theorems. An alternative approach to the quantification of entanglement is to rely on simpler mathematical expressions such as the 'tangle' defined earlier, which do not have any direct information-theoretic interpretation. This approach has been used by a number of authors (Barnett & Phoenix 1991; Jaeger *et al*. 1993, 1995; Shimony 1995; Schlienz & Mahler 1995, 1996; Linden & Popescu 1997; Grassl *et al*. 1997 Linden *et al*. 1998), and the following paragraphs are also written in this spirit.

We can gain significant insight into the nature of entanglement by studying the various entanglements that can exist among *three* objects. Let us therefore consider briefly the case of three spin-$\frac{1}{2}$ particles and explore some of the kinds of entanglement that arise there. (The three-particle case has also been studied in Shimony (1995), Schlienz & Mahler (1995, 1996), Linden & Popescu (1997), Grassl *et al*. (1997) and Linden *et al*. (1998).)

Consider a triple of spin-$\frac{1}{2}$ particles labelled A, B and C. The most general pure state of this system is

$$a_{000}|{\uparrow\uparrow\uparrow}\rangle + a_{001}|{\uparrow\uparrow\downarrow}\rangle + a_{010}|{\uparrow\downarrow\uparrow}\rangle + a_{011}|{\uparrow\downarrow\downarrow}\rangle \\ + a_{100}|{\downarrow\uparrow\uparrow}\rangle + a_{101}|{\downarrow\uparrow\downarrow}\rangle + a_{110}|{\downarrow\downarrow\uparrow}\rangle + a_{111}|{\downarrow\downarrow\downarrow}\rangle, \quad (4.7)$$

where the first symbol in each $|\dots\rangle$ refers to particle A, the second to B and the third to C. Any reasonable measure of entanglement should be invariant under separate rotations of the individual particles, a point that has been widely recognized and has been particularly developed in the papers of Schlienz & Mahler (1996), Linden & Popescu (1997), Grassl *et al*. (1997) and Linden *et al*. (1998). I would like to focus on three classes of invariants, all of which are generalizations of the determinant-like form appearing in equation (4.5), and all of which run from 0 (no entanglement) to 1 (maximum entanglement).

The first kind of invariant is the tangle between one of the particles and the other two, e.g. the tangle $\tau_{\mathrm{A(BC)}}$ between A and the pair BC. By choosing a suitable basis for BC, one can make the three-particle system look mathematically like a two-particle system, BC playing the role of one particle, and the tangle can then be defined as in equation (4.5). In terms of the original coefficients a_{ijk} this definition yields

$$\tau_{\mathrm{A(BC)}} = 2\left|\sum a_{imn}\bar{a}_{jmn}a_{i'pq}\bar{a}_{j'pq}\epsilon_{ii'}\epsilon_{jj'}\right|. \quad (4.8)$$

Here the bar indicates complex conjugation, and the sum is over all indices, each taking the values 0 and 1. The ϵ symbol is defined by $\epsilon_{01} = -\epsilon_{10} = 1$ and $\epsilon_{00} = \epsilon_{11} = 0$†. The quantities $\tau_{\mathrm{B(CA)}}$ and $\tau_{\mathrm{C(AB)}}$ can be expressed similarly.

† In the two-particle case, the tangle, defined in equation (4.5), can be written exactly as in equation (4.8) but with the last index on each a deleted.

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The second kind of invariant is the tangle between just two of the particles, ignoring the third. For example, τ_{AB} is the tangle between A and B. Now, particles A and B are typically in a mixed state since they are typically entangled with C, but equation (4.5) defines the tangle only for a pure state; so we need a new definition. We define τ_{AB} as the average tangle of the *pure* states of A and B that would result from a measurement on particle C, minimized over all possible complete measurements. An explicit formula for τ_{AB} can be found in the literature. (The tangle τ_{AB} is the square of what is called the 'concurrence' in Hill & Wootters (1997) and Wootters (1998).) In terms of the coefficients a_{ijk}, this formula is not as simple as the one for $\tau_{\mathrm{A(BC)}}$ and will not be given explicitly here, though it can be deduced from equation (4.10).

The third class of invariants consists of a single quantity which is symmetric under all permutations of the particles A, B and C and therefore measures a kind of entanglement that is shared among all three particles. Its formula is

$$\tau_{\mathrm{ABC}} = 2\left|\sum a_{ijk}a_{i'j'm}a_{npk'}a_{n'p'm'}\epsilon_{ii'}\epsilon_{jj'}\epsilon_{kk'}\epsilon_{mm'}\epsilon_{nn'}\epsilon_{pp'}\right|. \tag{4.9}$$

Even though this expression treats the third index differently from the other two, one finds that the quantity is indeed symmetric in the three particles.

There is an interesting relationship among these invariants, which we can think of as a simple mathematical law of entanglement:

$$\tau_{\mathrm{A(BC)}} = \tau_{\mathrm{AB}} + \tau_{\mathrm{AC}} + \tau_{\mathrm{ABC}}. \tag{4.10}$$

In words, the tangle of A with the rest of the system is equal to the tangle of A with B alone, plus the tangle of A with C alone, plus the essential three-way tangle of the whole system. To see how this works in a simple example, consider the state $(1/\sqrt{2})(|{\uparrow\uparrow\uparrow}\rangle + |{\downarrow\downarrow\downarrow}\rangle)$, known as the GHZ state (Greenberger *et al*. 1989). In this state, each particle is fully entangled with the rest of the system (e.g. $\tau_{\mathrm{A(BC)}} = 1$), and the three-way tangle τ_{ABC} is also equal to 1, but no two of the particles are entangled with each other: there is a classical up–down correlation between any two particles but no right–left correlation; so the particles are merely correlated, like two coins. Thus the above equation works out to read $1 = 0 + 0 + 1$. One sees that three-way entanglement can exist only at the expense of pairwise entanglement. The equation similarly shows us that entanglement between A and B can exist only at the expense of entanglement between A and C (the inequality $\tau_{\mathrm{A(BC)}} \geqslant \tau_{\mathrm{AB}} + \tau_{\mathrm{AC}}$ was proved by V. Coffman & J. Kundu (1998, unpublished work)). In other words, particle A has only a limited total capacity for entanglement, which can be expressed in three forms: entanglement with B, entanglement with C, and three-way entanglement of the triple. The fact that there are trade-offs among these entanglements makes explicit the specificity of connection mentioned earlier, which distinguishes entanglement from mere correlation: a coin can be completely correlated with each of 100 other coins, but a spin can be completely entangled with at most one other spin.

As with the information-theoretic measures of entanglement, there is much work yet to be done on these mathematical invariants (though much has already been done that I have not reported here). In particular, one wonders whether the kind of additivity exhibited in equation (4.10) generalizes to more complicated systems.

Even though entanglement has been studied in some ways for many decades, one has the feeling that we have hardly begun to appreciate either the rich mathematical structure of entanglement or the variety of applications that it allows. Further

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progress towards practical applications will depend on technological advances that make it easier to generate and control entangled objects. On the mathematical side, as the preceding paragraphs suggest, there are still many more unanswered questions about entanglement than answered ones. It will be exciting to see how experimental and theoretical developments will change our understanding and appreciation of entanglement over the next several years.

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Discussion

W. S. Leng (*London, UK*). Professor Wootters raised questions concerning the lack of any definition or laws of entanglement, and recalled an analogy with the laws of thermodynamics. Defining entanglement as symmetry-superpositions, and using this analogy, leads to the following laws of entanglement.

Law 1 (Conservation of energy). Conservation of entangling states in any entangling event. Given any distinct number of entanglement events, among any distinct number of objects to be entangled, then the total number of each value of the distinct possible entanglement-outcome states must be conserved.

Law 2 (Increase of entropy). Increase of entangling events in a closed system. The total number of entanglement events (in any closed set of entanglement objects) must always tend to increase.

Law 3 (Zero entropy). Equilibrium, stability under entanglements. Entanglement growth only vanishes under the condition that either there exists some specific condition or procedure preventing entanglements, or given already maximum possible entanglement events of entangling objects.

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Examples are teleportation, non-pure states, error growth, classical (pure) states, renormalization. These have predictive power, for example, law 1 under 3-entanglements (extend easily to n).

W. K. Wootters. If I understand correctly, all these laws are intended to apply to a closed system, from which entanglement cannot be removed by environmental interactions or by the change in one's state of knowledge that occurs when one performs a measurement. This is indeed an interesting case, and I agree that under these conditions entanglement tends to increase, if one defines entanglement as the sum of the entropies of the elementary objects composing the system. Moreover, this is a reasonable definition of entanglement: if the whole system is in a pure quantum state, then each component object can have non-zero entropy only by being entangled with something else.

I have two comments. First, there are other reasonable definitions of entanglement. In my paper I do not discuss how one should define the overall entanglement of a system of many objects, but this question is being pursued, and at this point there is no consensus on a definition. My second comment is that the particular sort of law I am thinking of is motivated by rather practical problems in which one does have to take into account the 'collapse' associated with measurement. (Entanglement distillation, for example, involves measurement.) Thus, even though in the many-worlds view the quantum state of the universe is getting more and more entangled, in our own experience we find that entanglement is a precious and delicate thing, easily destroyed by interactions or measurements. In other words, I am interested in laws of entanglement that apply to systems that are not necessarily closed.

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